Dynamic Adjustment of Double-Well Potential with Periodic k

1 Introduction

In this report, we investigate a system with a double-well potential where the asymmetry of the potential is controlled by a time-varying parameter $k(t)$. This type of potential is useful in understanding phenomena such as bistability and phase transitions, which can occur in systems that exhibit two stable equilibrium points under certain conditions.

2 General Form of the Potential

The most general form of the double-well potential with a time-varying control parameter $k(t)$ is given by:

$$V(x,t) = \frac{1}{4}ax^4 - \frac{1}{2}bx^2 + k(t)x$$

Where:

- a is a constant that controls the steepness of the potential's quartic term, which typically ensures the existence of two stable wells.
- b is a constant that affects the quadratic term, which also influences the shape and depth of the potential wells.
- $k(t)$ is a time-dependent control parameter that introduces an asymmetry into the potential, causing the relative depth of the two wells to change with time. The value of $k(t)$ can be periodic, leading to oscillations in the shape of the potential.

This potential represents a classical model for systems where phase transitions can occur as the system's energy landscape changes over time.

3 System Dynamics

The dynamics of a particle in this potential are governed by Newton's second law:

$$m\frac{d^2x}{dt^2} = -\frac{dV(x,t)}{dx}$$

Where $x(t)$ is the position of the particle, and m is the mass of the particle. The force acting on the particle is the negative gradient of the potential. Substituting the general potential into this equation, we get:

$$m\frac{d^2x}{dt^2} = -\left(ax^3 - bx + k(t)\right)$$

For simplicity, we typically set $m = 1$, resulting in the following equation of motion:

$$\frac{d^2x}{dt^2} = -(ax^3 - bx + k(t))$$

The system's behavior depends on the time-varying parameter $k(t)$, which can cause transitions between two stable wells or lead to a single stable state depending on the magnitude of $k(t)$.

4 Time-Varying $k(t)$

In this report, we consider $k(t)$ as a power law function to represent the dynamic adjustment of the potential. For instance, a simple periodic function such as:

$$k(t) = (t/10)^2$$

is used.

5 Simulation and Visualization

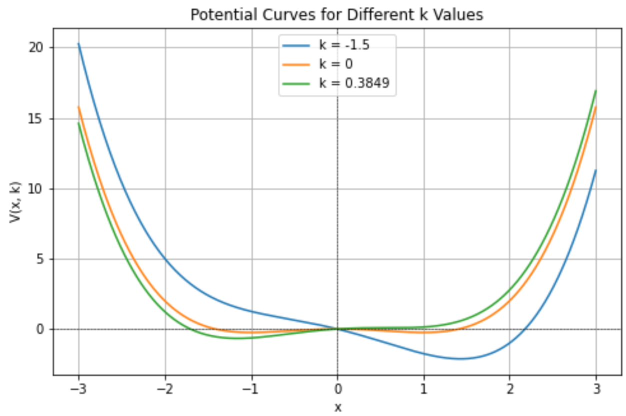
The following figures illustrate the time-varying potential and its effect on the system. We observe how the particle oscillates within the potential wells and how the wells themselves evolve as $k(t)$ changes.

5.1 Dynamic Potential

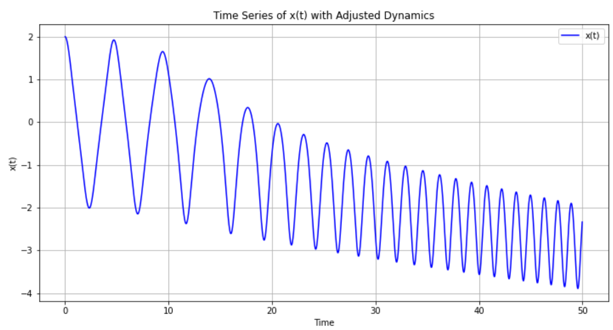
The potential $V(x,t)$ is shown at different time steps as $k(t)$ varies. The particle's position $x(t)$ is determined by the dynamics of the system as it oscillates in response to the changing potential.

5.2 Time Series of $x(t)$

The time series of $x(t)$ shows the oscillations of the particle within the potential. The particle's motion is influenced by the periodic changes in $k(t)$, causing it to oscillate between the wells and transition from bistability to monostability.



Dynamic Adjustment of the Double-Well Potential with Periodic $k(t)$.



Time Series of Particle Position $x(t)$ with Periodic $k(t)$.

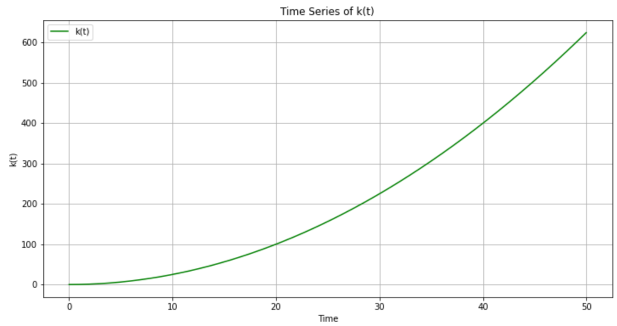
5.3 Value of $k(t)$ over Time

The plot of $k(t)$ shows its periodic variation, which introduces the oscillations in the potential. This variation plays a key role in determining the system's dynamics by altering the relative depth of the potential wells over time.

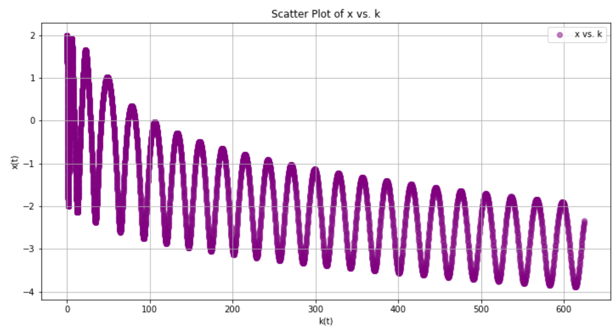
6 Critical k for the Potential Function

The potential function is given by:

$$V(x,k) = \frac{1}{4}bx^4 - \frac{1}{2}ax^2 + kx.$$



Time Series of the Control Parameter $k(t)$ Over Time.



Time Series of the Control Parameter $k(t)$ versus $x(t)$.

The first derivative of the potential is:

$$\frac{dV}{dx} = bx^3 - ax + k.$$

The critical values of k correspond to the point where the potential transitions from a double-well to a single-well. This occurs when the two minima of the potential merge, which is determined by the discriminant of the cubic equation:

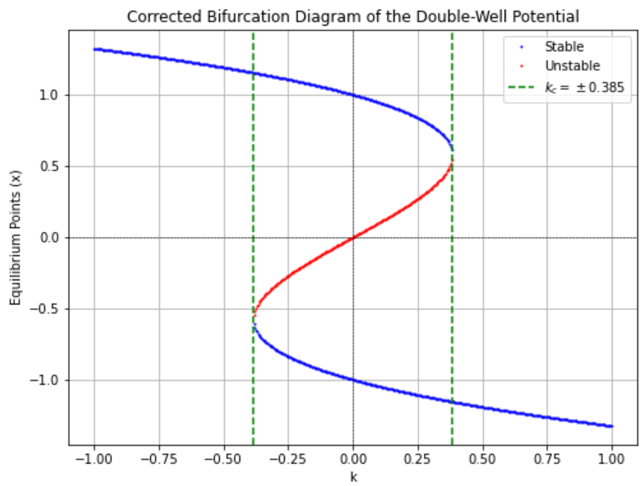
$$bx^3 - ax + k = 0.$$

The discriminant of the cubic equation is:

$$\Delta = b\left(4a^3 - 27bk^2\right).$$

The critical values of k are obtained by solving $\Delta = 0$, which yields:

$$k_c = \pm\frac{2\sqrt{3}}{9}\sqrt{\frac{a^3}{b}}.$$



Bifurcation Diagram for Double-Well Potential with Periodic k

Interpretation

- For $|k| < k_c$, the potential remains a double-well with two distinct minima.
- For $|k| \geq k_c$, the potential transitions to a single-well configuration.

Dynamics of the Double-Well Model

The dynamics of the system can be described by the following equation:

$$m\frac{d^2x}{dt^2} = -\frac{dV(x,k)}{dx} - \gamma\frac{dx}{dt} + \eta(t),$$

where:

- m is the mass, representing the inertia of the system.
- $-\frac{dV(x,k)}{dx}$ is the gradient of the potential, corresponding to the force acting on the system.
- $-\gamma\frac{dx}{dt}$ is the damping force, with γ being the damping coefficient. It is important for the transition from double stable to single stable state.
- $\eta(t)$ is an external noise term, modeled as Gaussian white noise with:

  $$\langle\eta(t)\rangle = 0, \quad \langle\eta(t)\eta(t')\rangle = 2D\delta(t-t'),$$

  where D is the noise intensity.

The potential function is given by:

$$V(x,k) = \frac{1}{4}ax^4 - \frac{1}{2}bx^2 + kx,$$

where $a > 0$ and $b > 0$ are coefficients, and k represents an external driving force.

The gradient of the potential is:

$$-\frac{dV(x,k)}{dx} = -ax^3 + bx - k.$$

Full Dynamic Equation

Substituting the gradient into the dynamic equation gives:

$$m\frac{d^2x}{dt^2} = -ax^3 + bx - k - \gamma\frac{dx}{dt} + \eta(t).$$

Overdamped Approximation

In the strong damping limit, where the inertia term can be neglected, the equation simplifies to a first-order stochastic differential equation:

$$\gamma\frac{dx}{dt} = -ax^3 + bx - k + \eta(t).$$

IN a more general case both a and b can also very with time, where

$$V(x,k) = \frac{1}{4}a(t)x^4 - \frac{1}{2}b(t)x^2 + k(t)x,$$

Applications

This model has a wide range of applications:

- **Physics**: Describing particle motion in a double-well potential.
- **Biology**: Modeling bistable switches, such as ion channel states.
- **Climate Science**: Analyzing transitions between stable climate states.
- **Economics**: Capturing decision-making processes with bistable dynamics.

7 Conclusion

This report demonstrates the use of a time-varying control parameter $k(t)$ in a double-well potential to simulate the dynamics of bistable systems. By varying $k(t)$ periodically, we observe transitions between two stable equilibrium points and a single stable state. The results provide insights into the behavior of systems with dynamic potentials and can be extended to more complex models in various fields, such as neuroscience, climate dynamics, and material science.